PERGAMON Computers and Mathematics with Applications 41 (2001) 597–606

An International Journal
computers & mathematics with applications

www.elsevier.nl/locate/camwa

Existence of Solutions for n^{th}-Order Integro-Differential Equations in Banach Spaces

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(Received November 1999; revised and accepted August 2000)

Abstract—This paper discusses the existence of solutions of initial value problems for n^{th}-order nonlinear integro-differential equations of mixed type on an infinite interval in a Banach space.

Keywords—Integro-differential equation, Banach space, Measure of noncompactness, Schauder fixed-point theorem.

1. INTRODUCTION

Nonlinear integro-differential equations arise from many nonlinear problems in science (see [1]). Papers [2–4] investigated the maximal and minimal solutions of periodic boundary value problems for first- and second-order nonlinear integro-differential equations of Volterra type and Hammerstein type in one-dimensional Euclidean space by means of the upper and lower solutions and monotone iterative technique. In a recent paper [5], we discussed the initial value problem (IVP) for second-order nonlinear integro-differential equations of mixed type on a finite interval in a real Banach space E by establishing a comparison result and using the monotone iterative technique. Now, in this paper, we shall investigate the existence of solutions of an IVP for n^{th}-order nonlinear integro-differential equations of mixed type on an infinite interval in E by means of a completely different method, that is, by using the fixed-point theory. In this situation, a new space $DC^{n-1}[J,E]$ is introduced and it is verified that the corresponding operator is completely continuous (i.e., continuous and compact). Consider the IVP for n^{th}-order nonlinear integro-differential equation of mixed type in E:

$$\begin{aligned} &u^{(n)} = f\left(t, u, u', \dots, u^{(n-1)}, Tu, Su\right), \qquad \forall\, t \in J, \\ &u(0) = u_0, \qquad u'(0) = u_1, \dots, u^{(n-1)}(0) = u_{n-1}, \end{aligned} \tag{1}$$

where $J = [0, \infty)$, $f \in C[J \times E \times E \times \cdots \times E, E]$, $u_i \in E (i = 0, 1, \dots, n-1)$ and

$$(Tu)(t) = \int_0^t k(t,s)u(s)\,ds, \qquad (Su)(t) = \int_0^\infty h(t,s)u(s)\,ds, \tag{2}$$

Research supported by the National Nature Science Foundation of China and the Research Fund for the Doctoral Program of Higher Education of China.
The author would like to thank the referee for valuable suggestions.

PII: S0898-1221(00)00303-5

$k \in C[D, R^1]$, $D = \{(t,s) \in J \times J : t \geq s\}$, $h \in C[J \times J, R^1]$, R^1 denotes the one-dimensional Euclidean space.

Let $BC[J,E] = \{u \in C[J,E] : e^{-t}\|u(t)\| \to 0 \text{ as } t \to \infty\}$ and $DC^{n-1}[J,E] = \{u \in C^{n-1}[J,E] : e^{-t}\|u^{(i)}(t)\| \to 0 \text{ as } t \to \infty, i = 0,1,\dots,n-1\}$, where $u^{(0)}(t) = u(t)$. It is easy to see that $BC[J,E]$ is a Banach space with norm

$$\|u\|_B = \max_{t\in J} \left(e^{-t}\|u(t)\|\right)$$

and $DC^{n-1}[J,E]$ is a Banach space with norm

$$\|u\|_D = \max\left\{\|u\|_B, \|u'\|_B, \dots, \left\|u^{(n-1)}\right\|_B\right\}.$$

$u \in C^n[J,E]$ is called a solution of IVP (1) if $u(t)$ satisfies (1).

2. SEVERAL LEMMAS

Let us list some conditions.

(H_1) $\lim_{t\to\infty}(e^{-t}\int_0^t |k(t,s)|e^s\,ds) = 0$, $\int_0^\infty |h(t,s)|e^s\,ds < \infty$ $(t \in J)$,

$$\lim_{t\to\infty}\left(e^{-t}\int_0^\infty |h(t,s)|e^s\,ds\right) = 0$$

and

$$\lim_{t'\to t}\int_0^\infty |h(t',s) - h(t,s)|\,e^s\,ds = 0, \qquad \forall\, t \in J = [0,\infty).$$

In this case, let

$$k^* = \max_{t\in J}\left(e^{-t}\int_0^t |k(t,s)|e^s\,ds\right), \qquad h^* = \max_{t\in J}\left(e^{-t}\int_0^\infty |h(t,s)|e^s\,ds\right).$$

(H_2) There exist $b, a_i \in C[J,R_+]$ $(i = 0,1,\dots,n+1)$ such that

$$\|f(t,v_0,v_1,\dots,v_{n-1},v_n,v_{n+1})\| \leq b(t) + \sum_{i=0}^{n+1} a_i(t)\,\|v_i\|,$$
$$\forall\, t \in J, \qquad v_i \in E, \quad (i = 0,1,\dots,n+1),$$

and

$$b^* = \int_0^\infty b(t)\,dt < \infty, \qquad a_i^* = \int_0^\infty a_i(t)e^t\,dt < \infty, \quad (i = 0,1,\dots,n+1),$$

where R_+ denotes the set of all nonnegative numbers.

(H_3) For any $r > 0$, f is uniformly continuous on $J_r \times B_r \times B_r \times \cdots \times B_r$ and $f(J_r, B_r, B_r, \dots, B_r)$ is relatively compact in E, where $J_r = [0,r]$ and $B_r = \{x \in E : \|x\| \leq r\}$.

REMARK 1. Condition (H_3) is satisfied automatically if E is finite dimensional.

LEMMA 1. *If Condition (H_1) is satisfied, then operator T and S defined by (2) are bounded linear operators from $BC[J,E]$ into $BC[J,E]$ and*

$$\|T\| \leq k^*, \qquad \|S\| \leq h^*. \tag{3}$$

PROOF. Equation (3) follows from the following inequalities:

$$e^{-t}\,\|(Tu)(t)\| \leq \left(e^{-t}\int_0^t |k(t,s)|e^s\,ds\right)\|u\|_B, \qquad \forall t \in J,$$

and

$$e^{-t}\|(Su)(t)\| \leq \left(e^{-t}\int_0^{\infty}|h(t,s)|e^s\,ds\right)\|u\|_B, \qquad \forall\, t \in J. \qquad \blacksquare$$

LEMMA 2. *Let Condition* (H_1) *be satisfied. Then,* $u \in C^n[J,E]$ *is a solution of IVP (1) if and only if* $u \in C^{n-1}[J,E]$ *and it is a solution of the following integral equation:*

$$\begin{aligned} u(t) = u_0 + tu_1 + \cdots + \frac{t^{n-1}}{(n-1)!}u_{n-1} + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} \\ f(s,u(s),u'(s),\dots,u^{(n-1)}(s),(Tu)(s),(Su)(s))\,ds, \qquad \forall\, t \in J. \end{aligned} \tag{4}$$

PROOF. By Taylor's formula, we have

$$\begin{aligned} u(t) = u(0) + tu'(0) + \cdots + \frac{t^{n-1}}{(n-1)!}u^{(n-1)}(0) + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}u^{(n)}(s)\,ds, \\ \forall\, u \in C^n[J,E]. \end{aligned} \tag{5}$$

So, if $u \in C^n[J,E]$ is a solution of IVP (1), then, substituting (1) into (5), we see that $u(t)$ satisfies (4). Conversely, if $u \in C^{n-1}[J,E]$ is a solution of (4), then direct differentiation of (4) gives

$$\begin{aligned} u'(t) &= u_1 + \cdots + \frac{t^{n-2}}{(n-2)!}u_{n-1} + \frac{1}{(n-2)!}\int_0^t (t-s)^{n-2} \\ &\quad \times f\left(s,u(s),u'(s),\dots,u^{(n-1)}(s),(Tu)(s),(Su)(s)\right)\,ds, \qquad \forall\, t \in J, \\ &\cdots \\ u^{(n-1)}(t) &= u_{n-1} + \int_0^t f\left(s,u(s),u'(s),\dots,u^{(n-1)}(s),(Tu)(s),(Su)(s)\right)\,ds, \qquad \forall\, t \in J, \\ u^{(n)}(t) &= f\left(t,u(t),u'(t),\dots,u^{(n-1)}(t),(Tu)(t),(Su)(t)\right), \qquad \forall\, t \in J, \end{aligned}$$

hence, $u \in C^n[J,E]$ and $u(t)$ is a solution of IVP (1). $\blacksquare$

Consider operator A defined by

$$\begin{aligned} (Au)(t) = u_0 + tu_1 + \cdots + \frac{t^{n-1}}{(n-1)!}u_{n-1} + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} \\ f\left(s,u(s),u'(s),\dots,u^{(n-1)}(s),(Tu)(s),(Su)(s)\right)\,ds. \end{aligned} \tag{6}$$

LEMMA 3. *If Conditions* (H_1) *and* (H_2) *are satisfied, then operator* A *defined by (6) is an operator from* $DC^{n-1}[J,E]$ *into* $DC^{n-1}[J,E]$ *and*

$$\|Au\|_D \leq M + \beta\|u\|_D, \qquad \forall\, u \in DC^{n-1}[J,E], \tag{7}$$

where

$$M = \max\{b^*, \|u_0\|, \|u_1\|, \dots, \|u_{n-1}\|\}, \qquad \beta = \sum_{i=0}^{n-1} a_i^* + k^*a_n^* + h^*a_{n+1}^*. \tag{8}$$

PROOF. For $u \in DC^{n-1}[J,E]$, we have

$$\begin{aligned} (Au)^{(i)}(t) = u_i + tu_{i+1} + \cdots + \frac{t^{n-i-1}}{(n-i-1)!}u_{n-1} + \frac{1}{(n-i-1)!}\int_0^t (t-s)^{n-i-1} \\ f\left(s,u(s),u'(s),\dots,u^{(n-1)}(s),(Tu)(s),(Su)(s)\right)\,ds, \qquad \forall\, t \in J\ (i = 0,1,\dots,n-1), \end{aligned} \tag{9}$$

so,

$$
\begin{aligned}
e^{-t}\left\|(Au)^{(i)}(t)\right\| &\le e^{-t}\left(1+t+\cdots+\frac{t^{n-i-1}}{(n-i-1)!}\right)\max\left\{\|u_i\|,\|u_{i+1}\|,\ldots,\|u_{n-1}\|\right\} \\
&+e^{-t}\frac{t^{n-i-1}}{(n-i-1)!}\int_0^t\left\|f\left(s,u(s),u'(s),\ldots,u^{(n-1)}(s),(Tu)(s),(Su)(s)\right)\right\|\,ds, \\
&\forall t\in J\ (i=0,1,\ldots,n-1).
\end{aligned}
\tag{10}
$$

On the other hand, Condition (H_2) and Lemma 1 imply that

$$
\begin{aligned}
&\left\|f\left(t,u(t),u'(t),\ldots,u^{(n-1)}(t),(Tu)(t),(Su)(t)\right)\right\| \\
&\le b(t)+\left(\sum_{i=0}^{n-1}a_i(t)+k^*a_n(t)+h^*a_{n+1}(t)\right)e^t\|u\|_D, \qquad \forall t\in J.
\end{aligned}
\tag{11}
$$

So,

$$
\int_0^t\left\|f\left(s,u(s),u'(s),\ldots,u^{(n-1)}(s),(Tu)(s),(Su)(s)\right)\right\|\,ds\le b^*+\beta\|u\|_D, \qquad \forall t\in J. \tag{12}
$$

It follows from (10) and (12) that

$$
\begin{aligned}
e^{-t}\left\|(Au)^{(i)}(t)\right\| &\le e^{-t}\left(1+t+\cdots+\frac{t^{n-i-1}}{(n-i-1)!}\right)\max\left\{\|u_i\|,\|u_{i+1}\|,\ldots,\|u_{n-1}\|\right\} \\
&+e^{-t}\frac{t^{n-i-1}}{(n-i-1)!}\left(b^*+\beta\|u\|_D\right), \qquad \forall t\in J, \quad (i=0,1,\ldots,n-1),
\end{aligned}
\tag{13}
$$

and therefore,

$$
e^{-t}\left\|(Au)^{(i)}(t)\right\|\le M+\beta\|u\|_D, \qquad \forall t\in J, \quad (i=0,1,\ldots,n-1). \tag{14}
$$

Now, (13) implies that $e^{-t}\|(Au)^{(i)}(t)\|\to 0$ as $t\to\infty$ $(i=0,1,\ldots,n-1)$, hence, $Au\in DC^{n-1}[J,E]$. Finally, (7) follows directly from (14). ∎

LEMMA 4. *Let Conditions (H_1) and (H_2) be satisfied and H be a bounded set in $DC^{n-1}[J,E]$. Then we have the following.*

(a) *$e^{-t}\|(Au)^{(i)}(t)\|\to 0$ as $t\to\infty$ uniformly in $u\in H$ $(i=0,1,\ldots,n-1)$.*
(b) *For any $r>0$, $(Au)^{(n-1)}(t)$ is equicontinuous on $t\in J_r=[0,r]$ for $u\in H$.*

PROOF. It is clear, (a) follows from (13). For $u\in H$, we have, by (9),

$$
(Au)^{(n-1)}(t)=u_{n-1}+\int_0^t f\left(s,u(s),u'(s),\ldots,u^{(n-1)}(s),(Tu)(s),(Su)(s)\right)ds, \qquad \forall t\in J. \tag{15}
$$

From (11), we see that $\|f(t,u(t),u'(t),\ldots,u^{(n-1)}(t),(Tu)(t),(Su)(t))\|$ is bounded for $u\in H$ and $t\in J_r$, so (15) implies that $(Au)^{(n-1)}(t)$ is equicontinuous on J_r for $u\in H$. ∎

In what follows, α and α_D denote the Kuratowskii measures of noncompactness of bounded sets in E and in $DC^{n-1}[J,E]$, respectively, (for measures of noncompactness, see [6–8]). For $H\subset DC^{n-1}[J,E]$, we write $H^{(i)}(t)=\{u^{(i)}(t):u\in H\}\subset E\,(t\in J, i=0,1,\ldots,n-1)$.

LEMMA 5. *Let H be a bounded set in $DC^{n-1}[J,E]$. Then*

$$
e^{-t}\alpha\left(H^{(i)}(t)\right)\le\alpha_D(H), \qquad \forall t\in J, \quad (i=0,1,\ldots,n-1), \tag{16}
$$

and, when Condition (H_1) *is satisfied,*

$$e^{-t}\alpha((TH)(t)) \le k^*\alpha_D(H), \quad e^{-t}\alpha((SH)(t)) \le h^*\alpha_D(H), \qquad \forall\, t \in J. \tag{17}$$

PROOF. For any $\epsilon > 0$, there exists a partition $H = \bigcup_{j=1}^{m} V_j$ such that $\operatorname{diam}(V_j) < \alpha_D(H) + \epsilon$ $(j = 1,2,\dots,m)$. Obviously, $H^{(i)}(t) = \bigcup_{j=1}^{m} V_j^{(i)}(t)$ $(t \in J, i = 0,1,\dots,n-1)$. For $u, v \in V_j$, $t \in J$, we have

$$e^{-t}\left\|u^{(i)}(t) - v^{(i)}(t)\right\| \le \operatorname{diam}(V_j) < \alpha_D(H) + \epsilon,$$

so,

$$\operatorname{diam}\left(V_j^{(i)}(t)\right) \le e^t\left(\alpha_D(H) + \epsilon\right), \qquad (j = 1,2,\dots,m),$$

and therefore, $\alpha(H^{(i)}(t)) \le e^t(\alpha_D(H) + \epsilon)$, which implies (16) by virtue of the arbitrariness of ϵ.

Since H is a bounded set in $DC^{n-1}[J,E]$, $H'(t)$ is bounded on $t \in J_r$, so $H(t)$ is equicontinuous on J_r, and therefore, (16) and [2, Theorem 1.2.2] imply that

$$\begin{aligned} e^{-t}\alpha((TH)(t)) &\le e^{-t}\int_0^t |k(t,s)|\alpha(H(s))\,ds \le \left(e^{-t}\int_0^t |k(t,s)|e^s\,ds\right)\alpha_D(H) \\ &\le k^*\alpha_D(H), \qquad \forall\, t \in J, \end{aligned} \tag{18}$$

and

$$\begin{aligned} e^{-t}\alpha((SH)(t)) &\le e^{-t}\int_0^r |h(t,s)|\alpha(H(s))\,ds + e^{-t}\alpha\left(\int_r^\infty h(t,s)H(s)\,ds\right) \\ &\le \left(e^{-t}\int_0^r |h(t,s)|e^s\,ds\right)\alpha_D(H) + e^{-t}\alpha\left(\int_r^\infty h(t,s)H(s)\,ds\right), \qquad \forall\, t \in J, \quad r > 0. \end{aligned} \tag{19}$$

Let $\|u\|_D \le M_0$ for $u \in H$. Then

$$\left\|\int_r^\infty h(t,s)u(s)\,ds\right\| \le \left(\int_r^\infty |h(t,s)|e^s\,ds\right)\|u\|_D \le M_0\left(\int_r^\infty |h(t,s)|e^s\,ds\right), \qquad \forall\, u \in H,$$

so,

$$\alpha\left(\int_r^\infty h(t,s)H(s)\,ds\right) \le 2M_0\left(\int_r^\infty |h(t,s)|e^s\,ds\right), \qquad \forall\, t \in J, \quad r > 0. \tag{20}$$

Now, observing (20) and taking limits as $r \to \infty$ in (19), we get

$$e^{-t}\alpha((SH)(t)) \le \left(e^{-t}\int_0^\infty |h(t,s)|e^s\,ds\right)\alpha_D(H) \le h^*\alpha_D(H), \qquad \forall\, t \in J. \tag{21}$$

Hence, (17) holds because of (18) and (21). ∎

LEMMA 6. *Let H be a bounded set in $DC^{n-1}[J,E]$. Suppose that $H^{(n-1)}(t)$ is equicontinuous on $J_r = [0,r]$ for any $r > 0$ and $e^{-t}\|u^{(i)}(t)\| \to 0$ as $t \to \infty$ uniformly for $u \in H$ $(i = 0,1,\dots,n-1)$. Then*

$$\alpha_D(H) = \max\left\{\sup_{t\in J}\left[e^{-t}\alpha\left(H^{(i)}(t)\right)\right] : i = 0,1,\dots,n-1\right\}. \tag{22}$$

PROOF. By Lemma 5, we have

$$\alpha_D(H) \ge \max\left\{\sup_{t\in J}\left[e^{-t}\alpha\left(H^{(i)}(t)\right)\right] : i = 0,1,\dots,n-1\right\}. \tag{23}$$

Let $\epsilon > 0$ be arbitrarily given. By hypotheses, there is an $r > 0$ such that

$$e^{-t}\left\|u^{(i)}(t)\right\| < \frac{\epsilon}{2}, \qquad \forall\, u \in H, \quad r \le t < \infty, \quad (i = 0,1,\dots,n-1). \tag{24}$$

Since H is a bounded set in $DC^{n-1}[J,E]$, $H^{(i)}(t)$ $(i=0,1,\ldots,n-1)$ are bounded on $t\in J_r$, so $H^{(i)}(t)$ $(i=0,1,\ldots,n-2)$ are equicontinuous on $t\in J_r$, together with the equicontinuity of $H^{(n-1)}(t)$ on J_r, we see that there exists a partition $0=t_0<t_1<\cdots<t_k<\cdots<t_p=r$ such that

$$\begin{gathered}\left\|e^{-t}u^{(i)}(t)-e^{-t_k}u^{(i)}(t_k)\right\|<\epsilon,\qquad \forall\, u\in H,\quad t\in[t_k,t_{k+1}],\\ (k=0,1,\ldots,p-1;i=0,1,\ldots,n-1).\end{gathered}\tag{25}$$

Let $J^{(k)}=[t_k,t_{k+1}]$ $(k=0,1,\ldots,p-1)$ and $J^{(p)}=[t_p,\infty)$. It follows from (24) and (25) that

$$\begin{gathered}\left\|e^{-t}u^{(i)}(t)-e^{-t_k}u^{(i)}\left(t_k\right)\right\|<\epsilon,\qquad \forall\, u\in H,\quad t\in J^{(k)},\\ (k=0,1,\ldots,p;\ i=0,1,\ldots,n-1).\end{gathered}\tag{26}$$

Let $B=\bigcup_{i=0}^{n-1}\bigcup_{k=0}^{p}e^{-t_k}H^{(i)}(t_k)$. There is a division $B=\bigcup_{j=1}^{m}W_j$ such that

$$\operatorname{diam}W_j<\alpha(B)+\epsilon,\qquad (j=1,2,\ldots,m).\tag{27}$$

Let Y be the set of all mappings from $\{0,1,\ldots,p\}$ into $\{1,2,\ldots,m\}$ and $Z=\{\beta=(\beta_0,\beta_1,\ldots,\beta_{n-1}):\beta_i\in Y, i=0,1,\ldots,n-1\}$. It is clear that Y and Z are finite sets. For $\beta=(\beta_0,\beta_1,\ldots,\beta_{n-1})\in Z$, let

$$H_\beta=\left\{u\in H:e^{-t_k}u^{(i)}\left(t_k\right)\in W_{\beta_i(k)},\ i=0,1,\ldots,n-1;\ k=0,1,\ldots,p\right\}.$$

We have $H=\bigcup_{\beta\in Z}H_\beta$. For any $u,v\in H_\beta$ and $t\in J$, we have $t\in J^{(k)}$ for some $k\in\{0,1,\ldots,p\}$, and so, (26) and (27) imply that

$$\begin{aligned}e^{-t}\left\|u^{(i)}(t)-v^{(i)}(t)\right\|&\le\left\|e^{-t}u^{(i)}(t)-e^{-t_k}u^{(i)}\left(t_k\right)\right\|+\left\|e^{-t_k}u^{(i)}\left(t_k\right)-e^{-t_k}v^{(i)}\left(t_k\right)\right\|\\&\quad+\left\|e^{-t_k}v^{(i)}\left(t_k\right)-e^{-t}v^{(i)}(t)\right\|<\alpha(B)+3\epsilon,\qquad (i=0,1,\ldots,n-1).\end{aligned}$$

Consequently,

$$\operatorname{diam}H_\beta\le\alpha(B)+3\epsilon,\qquad \forall\,\beta\in Z,$$

which implies $\alpha_D(H)\le\alpha(B)+3\epsilon$. Since $\epsilon>0$ is arbitrary, we get

$$\begin{aligned}\alpha_D(H)\le\alpha(B)&=\max\left\{e^{-t_k}\alpha\left(H^{(i)}(t_k)\right):k=0,1,\ldots,p;i=0,1,\ldots,n-1\right\}\\&\le\max\left\{\sup_{t\in J}\left[e^{-t}\alpha\left(H^{(i)}(t)\right)\right]:i=0,1,\ldots,n-1\right\}.\end{aligned}\tag{28}$$

Finally, (22) follows from (23) and (28). ∎

3. MAIN THEOREM

THEOREM 1. *Let Conditions (H_1)–(H_3) be satisfied. Assume that*

$$\beta=\sum_{i=0}^{n-1}a_i^*+k^*a_n^*+h^*a_{n+1}^*<1.\tag{29}$$

Then IVP (1) has a solution $u\in C^n[J,E]$ satisfying $e^{-t}\|u^{(i)}(t)\|\to 0$ as $t\to\infty$ for $i=0,1,\ldots,n-1$.

PROOF. By Lemmas 2 and 3, we know that operator A defined by (6) maps $DC^{n-1}[J,E]$ into $DC^{n-1}[J,E]$ and we need only to prove that A has a fixed point in $DC^{n-1}[J,E]$.

We first show that A is completely continuous, i.e., A is continuous and $A(H)$ is relatively compact in $DC^{n-1}[J,E]$ for any bounded set H in $DC^{n-1}[J,E]$. Let $u_m, u \in DC^{n-1}[J,E]$, $\|u_m - u\|_D \to 0$ $(m \to \infty)$. By (9), we have

$$\begin{gathered}e^{-t}\left\|\left(Au_m\right)^{(i)}(t) - (Au)^{(i)}(t)\right\| \le e^{-t}\frac{t^{n-i-1}}{(n-i-1)!}\int_0^t \Big\| f\Big(s, u_m(s), u_m'(s), \\ \dots, u_m^{(n-1)}(s), (Tu_m)(s), (Su_m)(s)\Big) \\ -f\Big(s, u(s), u'(s), \dots, u^{(n-1)}(s), (Tu)(s), (Su)(s)\Big)\Big\|\, ds, \\ \forall\, t \in J, \quad (i = 0, 1, \dots, n-1),\end{gathered} \tag{30}$$

so, (12) implies

$$\begin{gathered}e^{-t}\left\|\left(Au_m\right)^{(i)}(t) - (Au)^{(i)}(t)\right\| \le e^{-t}\frac{t^{n-i-1}}{(n-i-1)!}\left[2b^* + \beta\left(\|u_m\|_D + \|u\|_D\right)\right], \\ \forall\, t \in J, \quad (i = 0, 1, \dots, n-1).\end{gathered} \tag{31}$$

Let $\|u_m\| \le \gamma\,(m = 1, 2, 3, \dots)$ and $\epsilon > 0$ be arbitrarily given. By (31), we can choose a sufficiently large $r_0 > 0$ such that

$$e^{-t}\left\|\left(Au_m\right)^{(i)}(t) - (Au)^{(i)}(t)\right\| < \epsilon, \qquad \forall\, r_0 \le t < \infty, \quad (i = 0, 1, \dots, n-1). \tag{32}$$

Since $u_m^{(i)}(t) \to u^{(i)}(t)$ as $m \to \infty$ uniformly on $t \in J_r = [0, r]$ for any $r > 0$ $(i = 0, 1, \dots, n-1)$ and, by Lemma 1, $(Tu_m)(t) \to (Tu)(t)$ and $(Su_m)(t) \to (Su)(t)$ as $m \to \infty$ uniformly on $t \in J_r$ for any $r > 0$, we see from (H_3) and (30) that there is a positive integer m_0 such that

$$e^{-t}\left\|\left(Au_m\right)^{(i)}(t) - (Au)^{(i)}(t)\right\| < \epsilon, \qquad \forall\, m \ge m_0, \quad t \in [0, r_0], \quad (i = 0, 1, \dots, n-1). \tag{33}$$

It follows from (32) and (33) that $\|Au_m - Au\|_D \le \epsilon$ for $m \ge m_0$, hence, $\|Au_m - Au\|_D \to 0$ $(m \to \infty)$, and the continuity of A is proved. Let H be a bounded set in $DC^{n-1}[J,E]$. By Lemma 3, $A(H)$ is a bounded set in $DC^{n-1}[J,E]$. Observing Lemma 4 and using Lemma 6 to $A(H)$, we get

$$\alpha_D(A(H)) = \max\left\{\sup_{t\in J}\left[e^{-t}\alpha((A(H))^{(i)}(t))\right] : i = 0, 1, \dots, n-1\right\}. \tag{34}$$

On the other hand, from the equality

$$\int_0^a x(s)\, ds = \lim_{m\to\infty}\sum_{i=1}^m \frac{a}{m}x\left(\frac{ia}{m}\right), \qquad \forall\, x \in C[J_a, E], \quad (J_a = [0, a]),$$

we get a formula

$$\int_0^a x(s)\, ds \in a\,\overline{\mathrm{co}}\left\{x(s) : s \in J_a\right\}, \tag{35}$$

where $\overline{\mathrm{co}}\{A\}$ denotes the closed convex hull of A, so, by (9), we obtain

$$\begin{gathered}\alpha\left(\left(A(H)\right)^{(i)}(t)\right) \le \frac{t^{n-i}}{(n-i-1)!} \\ \alpha\left(\overline{\mathrm{co}}\, f\left(J_t, H(J_t), H'(J_t), \dots, H^{(n-1)}(J_t), (T(H))(J_t), (S(H))(J_t)\right)\right) \\ = \frac{t^{n-i}}{(n-i-1)!}\alpha\left(f\left(J_t, H(J_t), H'(J_t), \dots, H^{(n-1)}(J_t), (T(H))(J_t), (S(H))(J_t)\right)\right), \\ \forall\, t \in J, \quad (i = 0, 1, \dots, n-1),\end{gathered} \tag{36}$$

where $J_t = [0,t]$, $H^{(i)}(J_t) = \{u^{(i)}(s) : u \in H, s \in J_t\}$ $(i = 0,1,\dots,n-1)$. Since, for fixed $t \in J$, $H^{(i)}(J_t)$ $(i = 0,1,\dots,n-1)$ and $(T(H))(J_t), (S(H))(J_t)$ are bounded sets in E, so (H_3) implies that

$$\alpha\left(f\left(J_t, H(J_t), H'(J_t), \dots, H^{(n-1)}(J_t), (T(H))(J_t), (S(H))(J_t)\right)\right) = 0, \qquad \forall t \in J. \quad (37)$$

It follows from (36) and (37) that

$$\alpha\left((A(H))^{(i)}(t)\right) = 0, \qquad \forall t \in J, \quad (i = 0,1,\dots,n-1),$$

which implies by virtue of (34) that $\alpha_D(A(H)) = 0$, i.e., $A(H)$ is a relatively compact set in $DC^{n-1}[J,E]$. Thus, we have proved that A is a completely continuous operator from $DC^{n-1}[J,E]$ into $DC^{n-1}[J,E]$.

Let $R = M(1-\beta)^{-1}$, where M and β are defined by (8). We have $R > 0$ because $\beta < 1$ because of (29). Let $D_R = \{u \in DC^{n-1}[J,E] : \|u\|_D \le R\}$. For $u \in D_R$, we have by virtue of (7), $g\|Au\|_D \le M + \beta R = R$, so, $A(D_R) \subset D_R$. Hence, the Schauder fixed-point theorem implies that A has a fixed point in D_R. ∎

EXAMPLE 1. Consider the infinite system of scalar third-order integro-differential equations

$$\begin{aligned} u_n''' &= \frac{e^{-2t}}{5n}\left[u_{n+1} - \left(1 - u_n' + u_{n+2}''\right)^{1/3}\right] - \frac{te^{-2t}}{n^2}\left(1 - \int_0^t \frac{u_n(s)\,ds}{1+ts}\right)^{1/5} \\ &\quad + \frac{e^{-3t}}{2\sqrt{n}}\int_0^\infty e^{-2s}\cos(t-s)u_{2n}(s)\,ds, \qquad \forall\, 0 \le t < \infty; \\ u_n(0) &= \frac{1}{n}, \quad u_n'(0) = 0, \quad u_n''(0) = \frac{1}{n^2}, \qquad (n = 1,2,3,\dots). \end{aligned} \quad (38)$$

CONCLUSION. *Infinite system (38) has a C^3 solution $\{u_n(t)\}$ such that $u_n(t) \to 0$ as $n \to \infty$ for $0 \le t < \infty$ and $e^{-t}\sup_n |u_n^{(i)}(t)| \to 0$ as $t \to \infty$ $(i = 0,1,2)$.*

PROOF. Let $J = [0,\infty)$, $E = c_0 = \{u = (u_1,\dots,u_n,\dots) : u_n \to 0\}$ with norm $\|u\| = \sup_n |u_n|$. Then infinite system (38) can be regarded as an IVP of the form (1) in E. In this situation, $k(t,s) = (1+ts)^{-1}$, $h(t,s) = e^{-2s}\cos(t-s)$, $u = (u_1,\dots,u_n,\dots)$, $v = (v_1,\dots,v_n,\dots)$, $w = (w_1,\dots,w_n,\dots)$, $x = (x_1,\dots,x_n,\dots)$, $y = (y_1,\dots,y_n,\dots)$, and $f = (f_1,\dots,f_n,\dots)$, in which

$$\begin{aligned} f_n(t,u,v,w,x,y) &= \frac{e^{-2t}}{5n}\left[u_{n+1} - (1 - v_n + w_{n+2})^{1/3}\right] \\ &\quad - \frac{te^{-2t}}{n^2}(1 - x_n)^{1/5} + \frac{e^{-3t}}{2\sqrt{n}}y_{2n}. \end{aligned} \quad (39)$$

It is clear, $f \in C[J \times E \times E \times E \times E \times E, E]$ and $u_0 = (1,\dots,1/n,\dots) \in E$, $u_1 = (0,\dots,0,\dots) \in E$, $u_2 = (1,\dots,1/n^2,\dots) \in E$. It is easy to see that

$$k^* = \sup_{t \ge 0}\left(e^{-t}\int_0^t |k(t,s)|e^s\,ds\right) \le \sup_{t \ge 0}\left(1 - e^{-t}\right) = 1,$$
$$h^* = \sup_{t \ge 0}\left(e^{-t}\int_0^\infty |h(t,s)|e^s\,ds\right) \le \sup_{t \ge 0} e^{-t} = 1,$$

and

$$\int_0^\infty |h(t',s) - h(t,s)|\,e^s\,ds \le \int_0^\infty |\cos(t'-s) - \cos(t-s)|\,e^{-s}\,ds$$
$$\le |t'-t|\int_0^\infty e^{-s}\,ds = |t'-t| \to 0, \qquad \text{as } t' \to t, \quad \forall t \in J,$$

so, Condition (H_1) is satisfied. On the other hand, by using a simple inequality

$$(1+z)^\gamma \le 1+\gamma z, \qquad \forall 0 \le z < \infty, \quad 0<\gamma<1,$$

we see from (39) that

$$\begin{aligned} |f_n(t,u,v,w,x,y)| &\le \frac{e^{-2t}}{5n}\left[\|u\| + (1+\|v\|+\|w\|)^{1/3}\right] + \frac{te^{-2t}}{n^2}(1+\|x\|)^{1/5} + \frac{e^{-3t}}{2\sqrt{n}}\|y\| \\ &\le \frac{e^{-2t}}{5n}\left(\|u\| + 1 + \frac{1}{3}\|v\| + \frac{1}{3}\|w\|\right) + \frac{te^{-2t}}{n^2}\left(1+\frac{1}{5}\|x\|\right) + \frac{e^{-3t}}{2\sqrt{n}}\|y\|, \\ &\qquad (n=1,2,3,\dots), \end{aligned} \tag{40}$$

and therefore,

$$\|f(t,u,v,w,x,y)\| \le b(t) + a_0(t)\|u\| + a_1(t)\|v\| + a_2(t)\|w\| + a_3(t)\|x\| + a_4(t)\|y\|,$$

where

$$\begin{aligned} b(t) &= \frac{e^{-2t}}{5} + te^{-2t}, \qquad a_0(t) = \frac{e^{-2t}}{5}, \qquad a_1(t) = a_2(t) = \frac{e^{-2t}}{15}, \\ a_3(t) &= \frac{te^{-2t}}{5}, \qquad a_4(t) = \frac{e^{-3t}}{2}, \end{aligned}$$

which implies

$$b^* = \frac{7}{20}, \qquad a_0^* = \frac{1}{5}, \qquad a_1^* = a_2^* = \frac{1}{15}, \qquad a_3^* = \frac{1}{5}, \qquad a_4^* = \frac{1}{4}.$$

Hence, Condition (H_2) is satisfied, and inequality (29) also holds because

$$\beta = a_0^* + a_1^* + a_2^* + k^* a_3^* + h^* a_4^* \le \frac{47}{60} < 1.$$

Finally, we check Condition (H_3). Let $r>0$ be arbitrarily given. It is clear, f is uniformly continuous on $J_r \times B_r \times B_r \times B_r \times B_r \times B_r$. Let $\{t^{(m)}\} \subset J_r$, $\{u^{(m)}\} \subset B_r$, $\{v^{(m)}\} \subset B_r$, $\{w^{(m)}\} \subset B_r$, $\{x^{(m)}\} \subset B_r$, $\{y^{(m)}\} \subset B_r$. By virtue of (40), we have

$$\left|f_n\left(t^{(m)}, u^{(m)}, v^{(m)}, w^{(m)}, x^{(m)}, y^{(m)}\right)\right| \le \frac{1}{5n}\left(1+\frac{5r}{3}\right) + \frac{1}{2n^2}\left(1+\frac{r}{5}\right) + \frac{r}{2\sqrt{n}}, \quad (n,m=1,2,3,\dots), \tag{41}$$

therefore, $\{f_n(t^{(m)}, u^{(m)}, v^{(m)}, w^{(m)}, x^{(m)}, y^{(m)})\}$ is bounded, and so, by diagonal method, we can choose a subsequence $\{m_i\} \subset \{m\}$ such that

$$f_n\left(t^{(m_i)}, u^{(m_i)}, v^{(m_i)}, w^{(m_i)}, x^{(m_i)}, y^{(m_i)}\right) \to z_n, \qquad \text{as } i\to\infty, \quad (n=1,2,3,\dots). \tag{42}$$

Now, (41) and (42) imply

$$|z_n| \le \frac{1}{5n}\left(1+\frac{5r}{3}\right) + \frac{1}{2n^2}\left(1+\frac{r}{5}\right) + \frac{r}{2\sqrt{n}}, \qquad (n=1,2,3,\dots), \tag{43}$$

so $z=(z_1,\dots,z_n,\dots) \in c_0 = E$, and it is easy to see from (41)–(43) that

$$\begin{aligned} &\left\|f\left(t^{(m_i)}, u^{(m_i)}, v^{(m_i)}, w^{(m_i)}, x^{(m_i)}, y^{(m_i)}\right) - z\right\| \\ &= \sup_n \left|f_n(t^{(m_i)}, u^{(m_i)}, v^{(m_i)}, w^{(m_i)}, x^{(m_i)}, y^{(m_i)}) - z_n\right| \to 0, \qquad \text{as } i\to\infty. \end{aligned}$$

Thus, we have proved that $f(J_r, B_r, B_r, B_r, B_r, B_r)$ is relatively compact in E, and (H_3) is satisfied. Hence, our conclusion follows from Theorem 1. ∎

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